Applied Mathematics Letters 22 (2009) 856–859

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

New discrete type inequalities and global stability of nonlinear difference equations

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ARTICLE INFO

Article history:
Received 5 February 2008
Accepted 15 July 2008

Keywords:
Discrete type inequality
Halanay inequality
Global stability

ABSTRACT

In this work, we introduce discrete type inequalities. On the basis of these inequalities, we derive new global stability conditions for nonlinear difference equations.

1. Introduction

The investigation of stability and instability of nonlinear difference equations with delays has attracted a lot of attention from many researchers (see [3–6] and references cited therein). The Lyapunov stability theory is commonly used in order to obtain stability conditions. On the other hand, there are few stability conditions for nonlinear difference equations which are obtained by using discrete type inequalities. In [6], a generalized discrete Gronwall inequality is derived and new stability conditions for nonlinear difference equations with time delays are obtained. In [3], new stability conditions are given for nonlinear difference equations on the assumption that the nonlinear part satisfies certain discrete type inequalities. In [1,5], a discrete Halanay type inequality is introduced and used to study the discretized system of functional differential equations and to derive new stability conditions of nonlinear difference equations, respectively. Motivated by these results, we propose to give new discrete type inequalities and, on the basis of these inequalities, we derive new global stability conditions for nonlinear difference equations. This work is organized as follows. In Section 2, we introduce new discrete type inequalities. In Section 3, we derive global stability conditions by using inequalities obtained in Section 2 and show the advantages of our results, comparing with those in [3,4]. The conclusion is given in Section 4 and the work ends with cited references.

2. Discrete type inequalities

Let $\mathbb{R}$ denote the set of all real numbers; $\mathbb{R}^+$ the set of positive real numbers; $\mathbb{R}_0^+$ the set of nonnegative real numbers; $\mathbb{Z}$ the set of integers; and $\mathbb{Z}^+$ the set of positive integers; $\mathbb{Z}^{-r} = \{z \in \mathbb{Z} : z \geq -r\}$. Consider the following nonlinear difference equation:

$$\Delta x_n = f(n, x_n, x_{n-1}, \dots, x_{n-r}), \quad n \in \mathbb{Z}^+, \tag{2.1}$$

where $\Delta x_n = x_{n+1} - x_n$, and $f : \mathbb{N} \times \mathbb{R}^{r+1} \to \mathbb{R}$. According to [2,3], (2.1) is called a generalized difference equation and for each initial string $\{x_{-r}, x_{-r+1}, \dots, x_0\}$, (2.1) has a unique solution. In this section, we introduce new discrete type inequalities which will be used to derive global stability conditions in the next section.

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doi:10.1016/j.aml.2008.07.011

Theorem 2.1. *Let $q_i \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^+$, $i = 1, \ldots r$; $p, q_r \in \mathbb{R}^+$, where $0 = h_0 < h_1 < \cdots < h_r$ and $\sum_{i=0}^r q_i < p \le 1$, and let $\{x_j\}_{j \in \mathbb{Z}^{-h_r}}$ be a sequence of real numbers satisfying the inequality*

$$\Delta x_n \le -px_n + \sum_{i=0}^{r} q_i x_{n-h_i}, \quad n \in \mathbb{Z}^0. \tag{2.2}$$

Then there exists $\lambda_0 \in (0, 1)$ such that

$$x_n \le \max\{0, x_0, x_{-1}, \ldots, x_{-h_r}\}\lambda_0^n, \quad n \in \mathbb{Z}^0.$$

Moreover, λ_0 may be chosen as the smallest root of the polynomial

$$P(\lambda) = \lambda^{h_r+1} - (1 - p + q_0)\lambda^{h_r} - q_1\lambda^{h_r-h_1} \cdots - q_{r-1}\lambda^{h_r-h_{r-1}} - q_r \tag{2.3}$$

which lies in the interval $(0, 1)$.

Proof. Let $\{y_n\}$ be a solution of the difference equation

$$\Delta y_n = -py_n + \sum_{i=0}^{r} q_i y_{n-h_i}, \quad n \in \mathbb{Z}^0. \tag{2.4}$$

Since $q_i \in \mathbb{R}_0^+$ and $0 < p < 1$, it is straightforward to show that if $\{x_n\}$ satisfies (2.2) and $x_n \le y_n$ for $n = -h_r, \ldots, 0$, then $x_n \le y_n$ for all $n \in \mathbb{Z}^0$. For a given $K > 0$ and $\lambda \in (0, 1)$, the sequence $\{y_n\}$ defined by $y_n = K\lambda^n$ is a solution of Eq. (2.4) if and only if λ is a root of the polynomial (2.3). Since $\lim_{\lambda \to 0^+} P(\lambda) = -q_r < 0$ and $P(1) = p - \sum_{i=0}^r q_i > 0$, it follows from continuity of P that there exists a smallest real number $\lambda_0 \in (0, 1)$ such that $P(\lambda_0) = 0$. Thus, for any $K \in \mathbb{R}_0^+$, the sequence $\{K\lambda_0^n\}$ is a solution of (2.4). Let $K_0 = \max\{0, x_0, x_{-1}, \ldots, x_{-h_r}\}$. Then, $\{y_n\} = \left\{K_0\lambda_0^n\right\}$ is a solution of (2.4) and obviously we have $x_n \le y_n$, for $n = -h_r, \ldots, 0$. Therefore, by using the first part of the proof, we conclude that $x_n \le y_n = K_0\lambda_0^n$, $n \in \mathbb{Z}^0$. □

By an argument similar to that used in the proof of Theorem 2.1, we obtain the following result.

Theorem 2.2. *Let $p, \alpha_i, \beta_i \in \mathbb{R}^+$, $h_i \in \mathbb{Z}^+$, $i = 1, \ldots r$, where $0 = h_0 < h_1 < \cdots < h_r$, $\sum_{i=0}^r \alpha_i = 1$ and $\prod_{i=0}^r \beta_i < p \le 1$. Let $\{x_n\}_{n \in \mathbb{Z}^{-h_r}}$ be a sequence of real numbers such that $x_{n-h_i}^{\alpha_i}$ are defined for all $i = 1, \ldots r$; $n \in \mathbb{Z}^0$ which satisfies the inequality*

$$\Delta x_n \le -px_n + \prod_{i=0}^{r} \beta_i x_{n-h_i}^{\alpha_i}, \quad n \in \mathbb{Z}^0.$$

Then there exists $\lambda_0 \in (0, 1)$ such that

$$x_n \le \max\{0, x_0, x_{-1}, \ldots, x_{-h_r}\}\lambda_0^n, \quad n \in \mathbb{Z}^0.$$

Moreover, λ_0 may be chosen as the smallest root of the function

$$F(\lambda) = \lambda - \left(\prod_{i=0}^{r} \beta_i\right)\lambda^{-\sum_{i=1}^{r} h_i\alpha_i} + p - 1$$

which lies in the interval $(0, 1)$.

3. Global stability of nonlinear difference equations

In this section, we derive global stability conditions for nonlinear difference equation using discrete type inequalities derived in the previous section. Consider the nonlinear difference equation

$$\Delta x_n = -px_n + f(n, x_n, x_{n-h_1}, \ldots, x_{n-h_r}), \tag{3.1}$$

$n, h_i \in \mathbb{Z}^+$, $i = 1, \ldots r \in \mathbb{Z}^+$, $p > 0$. For any initial string $\{x_{-r}, x_{-r+1}, \ldots, x_0\}$, (3.1) has a unique solution which can be explicitly calculated. However, it is difficult to obtain stability conditions using that form of solution. The following result gives a global stability of solutions of (3.1) by using a discrete type inequality derived in Theorem 2.1.

Theorem 3.1. *Assume that there exist $q_i \in \mathbb{R}_0^+$, $h_i \in \mathbb{Z}^+$, $i = 1, \ldots r$; $q_r \in \mathbb{R}^+$, where $\sum_{i=0}^r q_i < p \le 1$ such that*

$$\left|f(n, x_n, x_{n-h_1}, \ldots, x_{n-h_r})\right| \le \sum_{i=0}^{r} q_i \left|x_{n-h_i}\right|, \tag{3.2}$$

for all $(n, x_n, x_{n-h_1}, \ldots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$. *Then, there exists* $\lambda_0 \in (0, 1)$ *such that every solution* $\{x_n\}$ *of* (3.1) *satisfies*

$$|x_n| \le \left(\max_{-h_r \le i \le 0}\{|x_i|\}\right)\lambda_0^n, \quad n \in \mathbb{Z}^0,$$

where λ_0 *is chosen as in* Theorem 2.1.

Proof. As in [2], it is straightforward to show that every solution $\{x_n\}$ of (3.1) can be written in the form

$$x_n = x_0(1-p)^n + \sum_{i=0}^{n-1}(1-p)^{n-i-1}f(i, x_i, x_{i-h_1}, \ldots, x_{i-h_r}), \quad n \in \mathbb{Z}^0.$$

By using (3.2), we obtain

$$|x_n| \le |x_0|(1-p)^n + \sum_{i=0}^{n-1}\sum_{j=0}^{r}(1-p)^{n-i-1}q_j\left|x_{i-h_j}\right|, \quad n \in \mathbb{Z}^0.$$

For each $n = -h_r, \ldots, 0$, let $v_n = |x_n|$ and for each $n \in \mathbb{Z}^+$, we let

$$v_n = |x_0|(1-p)^n + \sum_{i=0}^{n-1}\sum_{j=0}^{r}(1-p)^{n-i-1}q_j\left|x_{i-h_j}\right|.$$

Then, we have $|x_n| \le v_n, n \in \mathbb{Z}^{-h_r}$, and hence,

$$\Delta v_n = -pv_n + \sum_{i=0}^{r} q_i\left|x_{n-h_i}\right| \le -pv_n + \sum_{i=0}^{r} q_i v_{n-h_i}, \quad n \in \mathbb{Z}^0.$$

Therefore, by Theorem 2.1, we obtain

$$|x_n| \le v_n \le \left(\max_{-h_r \le i \le 0}\{v_i\}\right)\lambda_0^n = \left(\max_{-h_r \le i \le 0}\{|x_i|\}\right)\lambda_0^n, \quad n \in \mathbb{Z}^0,$$

where λ_0 is chosen as in Theorem 2.1. This completes the proof of the theorem. □

Similarly, by using Theorem 2.2 instead of Theorem 2.1, we obtain the following result.

Theorem 3.2. *Assume that there exist* $p, \alpha_i, \beta_i \in \mathbb{R}^+, h_i \in \mathbb{Z}^+, i = 1, \ldots r$, *where* $\sum_{i=0}^r \alpha_i = 1$ *and* $\prod_{i=0}^r \beta_i < p \le 1$. *such that*

$$\left|f(n, x_n, x_{n-h_1}, \ldots, x_{n-h_r})\right| \le \prod_{i=0}^{r}\beta_i\left|x_{n-h_i}\right|^{\alpha_i},$$

for all $(n, x_n, x_{n-h_1}, \ldots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$. *Then there exists* $\lambda_0 \in (0, 1)$ *such that every solution* $\{x_n\}$ *of* (3.1) *satisfies*

$$|x_n| \le \left(\max_{-h_r \le i \le 0}\{|x_i|\}\right)\lambda_0^n, \quad n \in \mathbb{Z}^0,$$

where λ_0 *is chosen as in* Theorem 2.2.

Remark 3.1. In [4], a discrete Halanay type inequality is given as in Theorem 2.1 where the inequality (2.2) is replaced by

$$\Delta x_n \le -px_n + q\max\{x_n, x_{n-1}, \ldots, x_{n-r}\}, \quad n \in \mathbb{Z}^0, \tag{3.3}$$

where $0 < q < p \le 1$. Note that if a sequence $\{x_n\}_{n\in\mathbb{Z}^{-r}}$ of positive real numbers satisfies (3.3), then it also satisfies (2.2). On the other hand, let $r = 1, p = \frac{5}{6}, q = q_0 = q_1 = \frac{1}{7}$; then we might easily show that the sequence $\{\frac{1}{2^n}\}_{n\in\mathbb{Z}^{-1}}$ satisfies (2.2) but not (3.3). Indeed,

$$\Delta x_n = \frac{1}{2^{n+1}} - \frac{1}{2^n} = -\frac{1}{2^{n+1}} \le -\frac{5}{6}\frac{1}{2^n} + \frac{1}{7}\left(\frac{1}{2^n} + \frac{1}{2^{n-1}}\right) = -\frac{17}{42}\frac{1}{2^n}.$$

On the other hand,

$$\Delta x_n = -\frac{1}{2^{n+1}} > -\frac{5}{6}\frac{1}{2^n} + \frac{1}{7}\max\left\{\frac{1}{2^n}, \frac{1}{2^{n-1}}\right\} = -\frac{23}{42}\frac{1}{2^n}.$$

Therefore, in the case of positive sequences, the discrete type inequality (2.2) is less conservative than the discrete Halanay type inequality given by (3.3).

Remark 3.2. In [3], it was shown that if $p = 1$ and

$$|f(n, x_n, x_{n-1}, \dots, x_{n-r})| \le \beta \prod_{i=0}^{r} |x_{n-i}|^{\alpha_i},$$

for all $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$, then (3.1) is locally asymptotically stable provided that either $\sum_{i=0}^{r} \alpha_i > 1$ and $\beta > 0$, or $\sum_{i=0}^{r} \alpha_i = 1$ and $\beta \in (0, 1)$. On the other hand, when $p = 1$, Theorem 3.2 ensures the global stability of the nonlinear difference equation (3.1) provided that $\sum_{i=0}^{r} \alpha_i = 1$ and $\beta \in (0, 1)$.

4. Conclusions

In this work, we introduce new discrete type inequalities and derive global stability conditions for nonlinear difference equations by using the inequalities obtained. We also show the advantages of our results, comparing to those in the previous literature.

Acknowledgments

This work was supported by the Thailand Research Fund and Commission for Higher Education, Thailand. The authors thank the anonymous reviewers and Prof. Ravi P. Agarwal for their valuable comments and suggestions which led to an improvement of the work.

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